Fermi National Accelerator Laboratory

FN-533

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March 15, 1990

Operated by Universities Research Association Inc. under contract with the United States Department of Energy

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ABSTRACT

A cylindrical waveguide of radius a is filled partially with an isotropic dielectric between radii $b < r < a$. A particle travels along the guide with an offset from the axis at a velocity $v < c$. The wake fields left behind are calculated. We find that the transverse wake forces do not vanish as $\gamma^{-2} = 1-(v/c)^2$ or in any other way when $v \to c$.

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I. INTRODUCTION

Wake field accelerators are of great interest because of their potential for providing a very high acceleration gradient for the next generation of accelerators. Best of all, it was shown experimentally that transverse deflections appeared to be small for dielectric-lined waveguides in contrast to the large transverse wake fields measured in structures and plasmas.[1] This is important because there will be small beam break up, which is a traditional source of beam instabilities in linear accelerators. A complete understanding of the transverse wake potential in such a dielectric-lined waveguide is therefore necessary. Recently, there have been some suggestions[2] that the transverse wake forces may vanish as γ^{-2} when the velocity v of the source particle approaches the velocity of light c, $[\gamma^{-2} = 1-(v/c)^2]$. The waveguide considered consists of a cylindrical metallic tube of radius a with infinite wall conductivity. The tube is filled partially with an isotropic material with dielectric constant ϵ between radii $b < r < a$. Our derivation with the same waveguide shows that the transverse wake forces do not vanish when $v \to c$.

II. SOLUTION

One way to solve for the wake fields is through the introduction of a scalar potential ϕ and a vector potential $\vec{A}$. In the Lorentz gauge, Maxwell's equations reduce to wave equations for ϕ and $\vec{A}$. The mathematics is rather complicated because the equations in $\vec{A}$ are coupled in the cylindrical coordinate.[3] The detail is given in the Appendix.

It is well-known that the transverse electric fields $\vec{E}_t$ and magnetic flux density $\vec{B}_t$ in a waveguide can always be expressed in terms of the longitudinal components E_z and B_z. In gaussian units, these relations are

$$\left(\nabla_z^2 - \frac{\mu\epsilon}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{B}_t = -\frac{i\omega\mu\epsilon}{c}\vec{\nabla}_t \times \hat{z}E_z + \vec{\nabla}_t\nabla_z B_z \ ,$$

$$\left(\nabla_z^2 - \frac{\mu\epsilon}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E}_t = \frac{i\omega}{c}\vec{\nabla}_t \times \hat{z}B_z + \vec{\nabla}_t\nabla_z E_z \ , \tag{2.1}$$

where μ and ϵ are respectively the relative magnetic permeability and dielectric constant of the medium under consideration.

In the presence of the dielectric, E_z and B_z are no longer independent. Thus there are two variables E_z and B_z to solve for. The problem is therefore much simpler than working with potentials.

The source particle carrying charge e travels with velocity $v = \beta c$ along the cylindrical waveguide at an offset r_0 from its axis. The Maxwell's equations for longitudinal fields are

$$\left(\nabla^2 - \frac{\mu\epsilon}{c^2}\frac{\partial^2}{\partial t^2}\right) E_z = \frac{4\pi}{\epsilon}\frac{\partial\rho}{\partial z} + \frac{4\pi\mu}{c^2}\frac{\partial J_z}{\partial t} \ , \tag{2.2}$$

$$\left(\nabla^2 - \frac{\mu\epsilon}{c^2}\frac{\partial^2}{\partial t^2}\right) B_z = 0 \ . \tag{2.3}$$

The charge density and current density are represented by, respectively,

$$\rho = e\frac{\delta(r-r_0)}{r}\delta(\theta)\delta(z-vt) \ , \tag{2.4}$$

$$J_z = v\rho \ , \tag{2.5}$$

where e is the charge of the particle under consideration. Note that all the above quantities are function of (r,θ,z,t). Let us denote the Fourier transforms in the variables $(z-vt)$ and θ by a tilde overhead, i.e.,

$$E_z(r,\theta,z,t) = \sum_{m=-\infty}^{\infty} \epsilon^{im\theta}\int_{-\infty}^{\infty} d\omega e^{i(z-vt)\omega/v}\tilde{E}_{zm}(r,\omega) \ . \tag{2.6}$$

With the help of

$$\delta(z-vt) = \frac{1}{2\pi v}\int_{-\infty}^{\infty} d\omega e^{i(z-vt)\omega/v} \ , \tag{2.7}$$

and

$$\frac{\delta(r-r_0)}{r}\delta(\theta) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im\theta}\int_0^{\infty} kdkJ_m(kr)J_m(kr_0) \ , \tag{2.8}$$

where J_m is the Bessel function of order m, Eq. (2.2) can be rewritten as

$$\left(\nabla^2 + \frac{\mu\epsilon\omega^2}{c^2}\right)\tilde{E}_{zm}(r,\omega) = \tilde{\phi}_m(r,\omega) \ , \tag{2.9}$$

where

$$\tilde{\phi}_m(r,\omega) = \frac{i4\pi\omega}{v\epsilon}\left(\tilde{\rho}_m - \frac{\mu\epsilon v}{c^2}\tilde{J}_m\right) = \frac{ie\omega(1-\mu\epsilon\beta^2)}{\pi v^2\epsilon}\int_0^{\infty} kdkJ_m(kr)J_m(kr_0) \ . \tag{2.10}$$

The particular solution can then be obtained easily as

$$\tilde{E}_{zm}^{\rm part} = -\frac{ie\omega}{\pi v^2\gamma^2}\int_0^{\infty} dk\frac{kJ_m(kr)J_m(kr_0)}{k^2+(\omega/v\gamma)^2} \ , \tag{2.11}$$

where μ and ϵ have been dropped because the particle is traveling in the vacuum sector, and $(1-v^2/c^2)$ has been replaced by γ^{-2}. The integration over k can be done exactly to become

$$\tilde{E}_{zm}^{\rm part} = -\frac{ie\omega}{\pi v^2\gamma^2}\begin{cases} I_m(\omega r/v\gamma)K_m(\omega r_0/v\gamma) & r<r_0 \\ \\ K_m(\omega r/v\gamma)I_m(\omega r_0/v\gamma) & r>r_0 \ , \end{cases} \tag{2.12}$$

where I_m and K_m are respectively modified Bessel function and Hankel function of order m.

For the general solution, we have

$$\tilde{E}_{zm}^{\rm gen} = \begin{cases} \mathcal{E}_m I_m(kr_0) I_m(kr) & 0 \le r \le b \\ \\ A_m[J_m(sa)Y_m(sr) - Y_m(sa)J_m(sr)] & b \le r \le a \ , \end{cases} \tag{2.13}$$

where

$$k = \frac{\omega}{v}\sqrt{1-\beta^2} \qquad \text{and} \qquad s = \frac{\omega}{v}\sqrt{\mu\epsilon\beta^2 - 1} \ , \tag{2.14}$$

and Y_m is Neuman function of order m. We have assumed that the dielectric constant is big enough so that $\mu\epsilon\beta^2 > 1$. Otherwise, there will not be any Cherenkov radiation produced and as a result there will not be any useful wake potential aside from space charge. In Eq. (2.14), the constants have been so chosen that E_z vanishes on the wall of the guide.

From Eq. (2.2), B_z can also be solved:

$$\tilde{B}_{zm}^{\rm gen} = \begin{cases} \mathcal{B}_m I_m(kr_0) I_m(kr) & 0 \le r \le b \\ \\ C_m[J_m'(sa)Y_m(sr) - Y_m'(sa)J_m(sr)] & b \le r \le a \ , \end{cases} \tag{2.15}$$

where arrangement has been made so that the radial component of $\vec{\tilde{B}}_m$ vanishes at the wall of the guide.

The four constants $\mathcal{E}_m$, $\mathcal{B}_m$, A_m, and C_m will be determined by matching boundary conditions at $r = b$, where the vacuum meets the dielectric. At $r = b$, E_z and B_z are

$$\begin{cases} \tilde{E}_{zm}^v = \mathcal{E}_m' I_m + \eta_m' K_m \\ \\ \tilde{E}_{zm}^d = A_m p_m \\ \\ \tilde{B}_{zm}^v = \mathcal{B}_m' I_m \\ \\ \tilde{B}_{zm}^d = C_m r_m \end{cases} \tag{2.16}$$

where the superscripts v and d denote vacuum and dielectric respectively. In the above the following abbreviations have been used:

$$
\begin{aligned}
I_m &= I_m(kb) \ , \\
K_m &= K_m(kb) \ , \\
p_m &= J_m(sa)Y_m(sb) - Y_m(sa)J_m(sb) \ , \\
r_m &= J'_m(sa)Y_m(sb) - Y'_m(sa)J_m(sb) \ , \\
\eta'_m &= \eta\frac{I_m(kr_0)}{\gamma^2} \ , \qquad \eta = -\frac{i\epsilon\omega}{\pi v^2} \ , \\
\mathcal{E}'_m &= \mathcal{E}_m I_m(kr_0) \ , \\
\mathcal{B}'_m &= \mathcal{B}_m I_m(kr_0) \ .
\end{aligned}
\tag{2.17}
$$

Knowing E_z and B_z, the transverse electric field can now be computed directly using Eq. (2.1). At the boundary $r = b$, these transverse components are

$$
\left\{
\begin{aligned}
\tilde{E}^v_{\theta m} &= \frac{iv\beta k}{\omega(1-\beta^2)}\mathcal{B}'_m I'_m + \frac{mv}{\omega b(1-\beta^2)}(\mathcal{E}'_m I_m + \eta'_m K_m) \\
\tilde{E}^d_{\theta m} &= -\frac{iv\beta s}{\omega(\mu\epsilon\beta^2-1)}C_m r'_m - \frac{mv}{\omega b(\mu\epsilon\beta^2-1)}A_m p_m \ ,
\end{aligned}
\right.
\tag{2.18}
$$

$$
\left\{
\begin{aligned}
\tilde{E}^v_{rm} &= \frac{mv\beta}{\omega b(1-\beta^2)}\mathcal{B}'_m I_m - \frac{ivk}{\omega(1-\beta^2)}(\mathcal{E}'_m I'_m + \eta'_m K'_m) \\
\tilde{E}^d_{rm} &= -\frac{mv\beta}{\omega b(\mu\epsilon\beta^2-1)}C_m r_m + \frac{ivs}{\omega(\mu\epsilon\beta^2-1)}A_m p'_m \ ,
\end{aligned}
\right.
\tag{2.19}
$$

where

$$
\begin{aligned}
I'_m &= I'_m(kb) \ , \\
K'_m &= K'_m(kb) \ , \\
p'_m &= J_m(sa)Y'_m(sb) - Y_m(sa)J'_m(sb) \ , \\
r'_m &= J'_m(sa)Y'_m(sb) - Y'_m(sa)J'_m(sb) \ .
\end{aligned}
\tag{2.20}
$$

All polarization charges and currents have been taken care of by the macroscopic magnetic permeability μ and dielectric constant ϵ. Therefore, there should be no surface charge or current at the boundary $r = b$. Thus we expect

$$
\tilde{E}^d_{zm} = \tilde{E}^v_{zm} \ ,
$$

$$
\begin{aligned}
\tilde{E}^d_{\theta m} &= \tilde{E}^v_{\theta m} \ , \\
\epsilon \tilde{E}^d_{rm} &= \tilde{E}^v_{rm} \ , \\
\tilde{B}^d_{zm} &= \mu \tilde{B}^v_{zm} \ .
\end{aligned} \tag{2.21}
$$

Then, the boundary conditions for B_θ and B_r will be satisfied automatically (since we have only 4 constants here). The four equations obtained from Eq. (2.21) are

$$A_m p_m = \mathcal{E}'_m I_m + \eta'_m K_m \ , \tag{2.22}$$

$$-\frac{i\beta s}{\mu\epsilon\beta^2-1} C_m r'_m - \frac{m}{b(\mu\epsilon\beta^2-1)} A_m p_m = i\beta k\gamma^2 \mathcal{B}'_m I'_m + \frac{m\gamma^2}{b}(\mathcal{E}'_m I_m + \eta'_m K_m) \ , \tag{2.23}$$

$$-\frac{m\beta\epsilon}{b(\mu\epsilon\beta^2-1)} C_m r_m + \frac{i\epsilon s}{\mu\epsilon\beta^2-1} A_m p'_m = \frac{m\beta\gamma^2}{b} \mathcal{B}'_m I_m - ik\gamma^2(\mathcal{E}'_m I'_m + \eta'_m K'_m) \ , \tag{2.24}$$

$$C_m r_m = \mu \mathcal{B}'_m I_m \ . \tag{2.25}$$

We solve for A_m from Eq. (2.22) and C_m from Eq. (2.25). Substituting into Eqs. (2.23) and (2.24), we get two equations for the two unknowns $\mathcal{E}'_m$ and $\mathcal{B}'_m$, the general solutions of the longitudinal electric field and magnetic flux density respectively:

$$
\begin{pmatrix}
(\mu\epsilon\beta^2-1)kbI'_m + \dfrac{\epsilon s b}{\gamma^2}\dfrac{p'_m I_m}{p_m} & m\beta(\mu\epsilon-1)I_m \\[2ex]
m\beta(\mu\epsilon-1)I_m & (\mu\epsilon\beta^2-1)kbI'_m + \dfrac{\mu s b}{\gamma^2}\dfrac{r'_m I_m}{r_m}
\end{pmatrix}
\begin{pmatrix}
\mathcal{E}'_m \\[2ex] i\mathcal{B}'_m
\end{pmatrix}
= -\eta'_m
\begin{pmatrix}
(\mu\epsilon\beta^2-1)kbK'_m + \dfrac{\epsilon s b}{\gamma^2}\dfrac{p'_m K_m}{p_m} \\[2ex]
m\beta(\mu\epsilon-1)K_m
\end{pmatrix} . \tag{2.26}
$$

III. THE MONOPOLE FIELDS

For the monopole case or $m = 0$, the square matrix on the left side of Eq. (2.26) is diagonal and the lower element of the right-hand matrix vanishes. We obtain immediately $\mathcal{B}'_0 = 0$; or there is no longitudinal magnetic field. The longitudinal electric field for the region $r_0 \le r \le b$ is

$$\tilde{E}_{z0} = \mathcal{E}'_0 I_0(kr) + \eta'_0 K_0(kr) \ , \tag{3.1}$$

where

$$\mathcal{E}'_0 = -\eta'_0 \frac{(\mu\epsilon\beta^2-1)kK'_0 + \gamma^{-2}\epsilon s p'_0 K_0/p_0}{(\mu\epsilon\beta^2-1)kI'_0 + \gamma^{-2}\epsilon s p'_0 I_0/p_0} \ . \tag{3.2}$$

For vanishingly small x,

$$I_0(x) \to 1 \ , \qquad xI_0'(x) \to \frac{x^2}{2} \ ,$$

$$K_0(x) \to -\ln\frac{|x|}{2} \ , \qquad xK_0'(x) \to -1 \ . \tag{3.3}$$

Therefore, $\tilde{E}_{z0}$ becomes

$$\tilde{E}_{z0} = \eta I_0(kr_0)\left[\frac{(\mu\epsilon-1)p_0}{p_0' + (sb/2\epsilon)p_0}\frac{I_0(kr)}{\epsilon sb} - \frac{K_0(kr)}{\gamma^2}\right] \ , \tag{3.4}$$

where η is given in Eq. (2.17)

The transverse forces on any charge e traveling with velocity v behind the source can be obtained from the Panofsky-Wenzel theorem, and are related to the longitudinal electric field by

$$\tilde{F}_{rm} = \frac{ev}{i\omega}\frac{\partial \tilde{E}_{zm}}{\partial r} \ , \tag{3.5}$$

$$\tilde{F}_{\theta m} = \frac{emv}{i\omega r}\tilde{E}_{zm} \ . \tag{3.6}$$

Although $\tilde{F}_{\theta 0} = 0$, it is evident that the radial transverse force $\tilde{F}_{r0}$ is not zero since $\tilde{E}_{z0}$ clearly depends on r through $I_0(kr)$ and $K_0(kr)$. However, since $k = \omega/\gamma v$, this dependence is very small at large γ. In fact, it can be easily shown that

$$\tilde{F}_{r0} = \mathcal{O}(\gamma^{-2}r) \qquad \text{or} \qquad \mathcal{O}(\gamma^{-2}r^{-1}) \ , \tag{3.7}$$

in the region $r_0 \le r \le b$, and vanishes when $\gamma \to \infty$. This reminds us of the behavior of the space-charge forces which also go to zero as γ^{-2}.

However, in the presence of a dielectric lining, the longitudinal electric field tends to a nonzero limit as $\gamma \to \infty$. Using Eqs. (2.6) and (3.4), we obtain

$$E_{z0}(r,z,t) = -\frac{ie\sqrt{\mu\epsilon-1}}{\pi\epsilon bc}\int_{-\infty}^{\infty} d\omega e^{i\omega(z-ct)/c}\frac{p_0}{p_0' + (sb/2\epsilon)p_0} \ , \tag{3.8}$$

with $s = \omega\sqrt{\mu\epsilon - 1}/c$. We next change the variable of integration to s and integrate in the complex s-plane. To satisfy causality, the poles of the integrand are placed slightly below the real s-axis. Since p_0 is even and $p_0' + (sb/2\epsilon)p_0$ is odd, we obtain for $z < ct$,

$$E_{z0}(r,z,t) = -\frac{4e}{\epsilon b}\sum_\lambda \frac{p_0}{\frac{d}{ds}[p_0' + (sb/2\epsilon)p_0]}\cos\frac{s(z-ct)}{\sqrt{\mu\epsilon-1}}\bigg|_{s=s_\lambda} \ , \tag{3.9}$$

where s_λ is the λ-th *positive* root of

$$p_0' + \frac{sb}{2\epsilon} p_0 = 0 \ . \tag{3.10}$$

This result is in complete agreement with that of Gai.[2] Lots of physics are embedded in Eq. (3.9). Cherenkov radiation is produced inside the dielectric at an angle $\sin^{-1}(1/\sqrt{\mu\epsilon}c)$ with the axis of the guide. Because the velocity of light in the dielectric $\hat{c} = c/\sqrt{\mu\epsilon}$ is less than the velocity of the source particle, this radiation bounces back and forth inside the dielectric layer and penetrates into the central vacuum region of the guide, creating a wake potential lagging behind. It is this potential that we hope would perform the required acceleration on other particles.

IV. HIGHER-MULTIPOLE FIELDS

For the higher multipole, i.e., $m \neq 0$, the longitudinal magnetic flux density is no longer zero. The square matrix in Eq. (2.26) is not diagonal, but both $\mathcal{E}_m'$ and $\mathcal{B}_m'$ can be solved easily. The result will be different from that of Gai.

When $m \neq 0$, Eq. (2.26) can be rewritten as

$$\begin{pmatrix} \dfrac{kbI_m'}{mI_m} + \dfrac{\epsilon k^2 b}{s} \dfrac{p_m'}{mp_m} & \dfrac{\beta(\mu\epsilon-1)}{\mu\epsilon\beta^2-1} \\ \dfrac{\beta(\mu\epsilon-1)}{\mu\epsilon\beta^2-1} & \dfrac{kbI_m'}{mI_m} + \dfrac{\mu k^2 b}{s} \dfrac{r_m'}{mr_m} \end{pmatrix} \begin{pmatrix} \mathcal{E}_m' \\ \\ i\mathcal{B}_m' \end{pmatrix}$$

$$= -\eta_m' \frac{K_m}{I_m} \begin{pmatrix} \dfrac{kbK_m'}{mK_m} + \dfrac{\epsilon k^2 b}{s} \dfrac{p_m'}{mp_m} \\ \\ \dfrac{\beta(\mu\epsilon-1)}{\mu\epsilon\beta^2-1} \end{pmatrix} , \tag{4.1}$$

where we have used the relation $(s/k)^2 = \gamma^2(\mu\epsilon\beta^2 - 1)$. Employing the small-argument expansions of the modified Bessel functions for $m \neq 0$:

$$I_m(x) = \frac{1}{m!}\left(\frac{x}{2}\right)^m \left[1 + \frac{1}{m-1}\left(\frac{x}{2}\right)^2\right] ,$$

$$xI_m'(x) = \frac{1}{(m-1)!}\left(\frac{x}{2}\right)^m \left[1 + \frac{m+2}{m(m+1)}\left(\frac{x}{2}\right)^2\right] ,$$

$$K_m(x) = \frac{(m-1)!}{2}\left(\frac{x}{2}\right)^{-m} \left[1 + \mathcal{O}\left(\frac{x}{2}\right)^2\right] ,$$

$$xK_m'(x) = -\frac{m!}{2}\left(\frac{x}{2}\right)^{-m}\left[1 - \mathcal{O}\left(\frac{x}{2}\right)^2\right] , \tag{4.2}$$

we obtain

$$\frac{kbI_m'}{mI_m} = 1 + \mathcal{O}\left(\gamma^{-2}\right) \qquad \text{and} \qquad \frac{kbK_m'}{mK_m} = -1 + \mathcal{O}\left(\gamma^{-2}\right) . \tag{4.3}$$

Therefore, Eq. (4.1) for $\mathcal{E}_m'$ and $\mathcal{B}_m'$ becomes

$$\begin{pmatrix} 1+a_1 & 1+a_2 \\ 1+a_2 & 1+a_3 \end{pmatrix}\begin{pmatrix} \mathcal{E}_m' \\ i\mathcal{B}_m' \end{pmatrix} = -\eta_m' \frac{K_m}{I_m}\begin{pmatrix} -1+a_4 \\ 1+a_2 \end{pmatrix} , \tag{4.4}$$

where a_1, a_2, a_3, and a_4 are $\mathcal{O}(\gamma^{-2})$. The 1's in the above matrix elements and the signs before them are extremely important. The four 1's in the square matrix lead to a near cancellation of the determinant, leaving behind

$$\det = (a_1 - 2a_2 + a_3) + (a_1 a_3 - 2a_2^2) , \tag{4.5}$$

which is $\mathcal{O}(\gamma^{-2})$. The -1 and $+1$ in the right-hand matrix, on the other hand, add when solving for $\mathcal{E}_m'$ and $\mathcal{B}_m'$. Keeping the lowest order contribution, we obtain simply

$$\mathcal{E}_m' = \frac{2\eta_m'}{\det}\frac{K_m}{I_m} . \tag{4.6}$$

The corresponding longitudinal electric field is therefore

$$\tilde{E}_{zm} = \frac{2\eta}{D(s)}\frac{K_m}{I_m} I_m(kr_0) I_m(kr) \rightarrow \frac{i2e\omega}{\pi c^2}\frac{1}{mD(s)}\left(\frac{r_0}{b}\right)^m\left(\frac{r}{b}\right)^m , \tag{4.7}$$

where

$$D(s) = \gamma^2 \times \det\Big|_{\gamma\to\infty} . \tag{4.8}$$

We see clearly that $\tilde{E}_{zm}$ and therefore the transverse forces do not vanish in the limit $\gamma \to \infty$.

Now, let us evaluate the determinant. Using the small-argument expansions of the modified Bessel functions in Eq. (4.2), we get

$$\gamma^2 a_1 = \frac{1}{\mu\epsilon - 1}\left[\frac{s^2 b^2}{2m(m+1)} + \frac{\epsilon s b p_m'}{m p_m}\right] ,$$

$$\gamma^2 a_2 = \frac{\mu\epsilon + 1}{2(\mu\epsilon - 1)} ,$$

$$\gamma^2 a_3 = \frac{1}{\mu\epsilon - 1}\left[\frac{s^2b^2}{2m(m+1)} + \frac{\mu s b r'_m}{m r_m}\right] . \tag{4.9}$$

From Eqs. (4.5) and (4.8), we have

$$D(s) = \frac{1}{\mu\epsilon - 1}\left[\frac{s^2b^2}{m(m-1)} + \frac{\epsilon s b p'_m}{m p_m} + \frac{\mu s b r'_m}{m r_m} - (\mu\epsilon + 1)\right] . \tag{4.10}$$

Substitute the results in Eq. (4.7) and then Eq. (2.6). The integration is then performed in the complex s plane to obtain for $z < ct$,

$$E_{zm}(r,z,t) = 8\epsilon\left(\frac{r_0}{b}\right)^m\left(\frac{r}{b}\right)^m \sum_\lambda \frac{s}{\frac{d}{ds}\mathcal{D}(s)} \cos\frac{s(z-ct)}{\sqrt{\mu\epsilon-1}}\bigg|_{s=s_\lambda} , \tag{4.11}$$

where s_λ is the λ-th *positive* root of

$$\mathcal{D}(s) = \frac{s^2b^2}{m+1} + \frac{\epsilon s b p'_m}{p_m} + \frac{\mu s b r'_m}{r_m} - m(\mu\epsilon + 1) = 0 . \tag{4.12}$$

Thus, for $m \neq 0$, E_{zm} does not vanish when $\gamma \to \infty$. The transverse wake forces can be obtained readily by Eqs. (3.5) and (3.6), and they also do not vanish as $\gamma \to \infty$.

V. DISCUSSIONS

(1) We see that for the higher-order multipoles, the longitudinal and transverse wake forces are of the same order of magnitude as the monopole longitudinal wake force with respect to γ^{-2}, in apparent contradiction to what was observed experimentally. Experimental observations were made using a source bunch having a gaussian extension with rms length σ_ℓ. If the center of the bunch travels according to $z = ct$, the wakes behind are given by Eq. (3.9) and (4.11) with each term in the summand multiplied by

$$\exp\left[-\frac{1}{2}\left(\frac{s_\lambda \sigma_\ell}{\sqrt{\mu\epsilon - 1}}\right)^2\right] . \tag{5.1}$$

Consequently, only the first one or two characteristic waves will contribute significantly. If the lowest wave numbers s_λ for higher-multipole wakes were very much larger than the lowest wave number of the monopole wake, the higher-multipole wakes and therefore the transverse forces would be suppressed. However, such a possibility is not encouraging. For a cylindrical waveguide without dielectric lining, the lowest propagating mode is TE_{10}, which is lower than TM_{00}. This leads us to believe that the dipole mode in the dielectric-lined waveguide has the lowest characteristic frequency.

(2) For a particle of charge e traveling with velocity $v\hat{z}$ in a TE field ($E_z = 0$), the transverse force can be easily expressed by using Eq. (2.1),

$$\begin{cases} F_r^{\mathrm{TE}} = -eB_\theta^{\mathrm{TE}}(\beta - \beta_p) \\ F_\theta^{\mathrm{TE}} = eB_r^{\mathrm{TE}}(\beta - \beta_p) \end{cases} . \tag{5.2}$$

If the electromagnetic field is purely TM ($B_z = 0$), the transverse force is

$$\vec{F}_t^{\mathrm{TM}} = e\vec{E}_t^{\mathrm{TM}}(1 - \beta_p\beta) . \tag{5.3}$$

In the above, $\beta_p = v_p/c$ and v_p is the phase velocity of the electromagnetic wave in the z-direction. There is a theorem which says that an electromagnetic wave can be written as a linear combination of a TM-wave, a TE-wave, and a TEM-wave. Our wake field inside the dielectric waveguide certainly obeys the theorem, but without TEM contribution. Therefore, if the test particle has a velocity v equal to the phase velocity v_p of the electromagnetic field, the TE part of the transverse force on the test particle will vanish according to Eq. (5.2), and the TM part of the transverse force will be suppressed by γ^2 according to Eq. (5.3). However, this suppression is cancelled by the fact that $\vec{E}_t^{\mathrm{TM}}$ in vacuum is of order γ^2 as illustrated by the part involving $\mathcal{E}'_m$ in Eqs. (2.18) or (2.19). If the electromagnetic fields inside the waveguide were not wake fields left by a particle but were excited by some other means, there would not be such enhancement of $\vec{E}_t^{\mathrm{TM}}$ and the γ^2 suppression of the transverse force would become realistic. An example is the dielectric particle separator designed by Chang and Dawson.[4]

However, it is also difficult to understand why the transverse fields can be of order γ^2. The Lorentz contracted fields of a particle is of order γ with an opening angle $\mathcal{O}(\gamma^{-1})$ so that the total flux is independent of γ. Here, instead of a small opening angle, the transverse fields are distributed longitudinally as a cosine function as illustrated by Eqs. (3.5), (3.6), and (4.11). The total transverse flux will blowup as γ^2. A closer examination of Eqs. (2.18) or (2.19) reveals that $\vec{E}_t^{\mathrm{TE}}$, the part involving $\mathcal{B}'_m$, is also of order γ^2. The contribution of the source, the part involving η'_m, is $\mathcal{O}(1)$. With the solution of $\mathcal{E}'_m$ and $\mathcal{B}'_m$ from Eq. (2.26), we find that the mystery is solved, because the γ^2 parts of $\vec{E}_t^{\mathrm{TE}}$ and $\vec{E}_t^{\mathrm{TM}}$ cancel exactly, leaving $\vec{E}_t$ finite as $\gamma \to \infty$.

The author would like to thank Drs. Eric Chojnacki, Wei Gai, Sekazi Mtingwa, Jim Rosenzweig, Mike Rosing, Paul Schoessow, and James Simpson for useful and encouraging discussions.

APPENDIX

We introduce a scalar potential ϕ and a vector potential $\vec{A}$, defined by

$$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} ,$$

$$\vec{B} = \vec{\nabla} \times \vec{A} . \tag{A.1}$$

In order to transform Maxwell's equations into independent inhomogeneous equations in ϕ for $\vec{A}$,

$$\nabla^2\phi - \frac{\mu\epsilon}{c^2}\frac{\partial^2\phi}{\partial t^2} = -\frac{4\pi\rho}{\epsilon} , \tag{A.2}$$

$$\nabla^2\vec{A} - \frac{\mu\epsilon}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\frac{4\pi\mu\vec{J}}{c} , \tag{A.3}$$

with $\vec{J} = \rho\vec{v}$, use is made of the Lorentz condition

$$\vec{\nabla}\cdot\vec{A} + \frac{\mu\epsilon}{c}\frac{\partial\phi}{\partial t} = 0 . \tag{A.4}$$

Note that in cylindrical coordinate,

$$\nabla^2\vec{A} = \hat{r}\left[\nabla^2 A_r - \frac{A_r}{r^2} - \frac{2}{r^2}\frac{\partial A_\theta}{\partial\theta}\right] + \hat{\theta}\left[\nabla^2 A_\theta - \frac{A_\theta}{r^2} + \frac{2}{r^2}\frac{\partial A_r}{\partial\theta}\right] + \hat{z}\nabla^2 A_z . \tag{A.5}$$

In other words, the equations for A_r and A_θ are coupled. Because of the symmetry of Eqs. (A.2) and (A.1), one is tempted to assign

$$\vec{A} = \frac{\vec{v}}{c}\mu\epsilon\phi . \tag{A.6}$$

This simplifies the problem tremendously because it leaves behind only one equation in one variable. However, Eq. (A.6) may not be correct. In fact, the relation between ϕ and $\vec{A}$ has been given explicitly by Eq. (A.4). Any additional constraint can arise only from the speciality of the problem. For example, Eq. (A.6) can be correct if (1) $(\vec{A},\phi)$ rotates as a 4-vector in the Minskowski space and (2) there is no longitudinal magnetic field (TM modes). But, in the presence of dielectric, space-time does not constitute a Minskowski space and the electromagnetic fields do not separate into pure TE or TM modes.

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